# Coherence and clustering in ensembles of neural networks

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## Abstract

Large ensembles of globally coupled chaotic neural networks undergo a transition to complete synchronization for high coupling intensities. The onset of this fully coherent behavior is preceded by a regime where clusters of networks with identical activity are spontaneously formed. In these regimes of coherent collective evolution the dynamics of each neural network is still chaotic. These results may be relevant to the study of systems where interaction between elements is able to give rise to coherent complex behavior, such as in cortex activity or in insect societies.

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Mutual synchronization may play a significant role in the emergence of coherent collective behavior of different biological populations [1]. Studies of globally coupled nonlinear oscillators show the possibility of spontaneous transitions to a coherent regime characterized by identical states of all elements [2-5]. Similar transitions, known as complete stochastic synchronization, have been found for globally coupled chaotic logistic maps [6], employed in the description of ecological dynamics [7], and for populations of several coupled chaotic neurons [8]. However, they deem improbable when larger and more complex systems are considered. In contrast to this view, our study reveals that complete mutual synchronization can easily be achieved even in ensembles whose members represent large neural networks with complex individual dynamics. In the emerging coherent regime, the signals generated by all networks in the ensemble are identical. The transition is preceded by clustering, where groups of neural networks with coherent activity patterns are spontaneously formed. Our finding may contribute towards better understanding of coherent activity in the cortex and in biological populations, such as swarms or insect societies. It can find practical applications in the design of robotic ensembles.

Real neurons are complicated and, in theoretical studies of collective neural activity, greatly simplified models are used [9-11]. To investigate synchronization in ensembles of coupled neural networks, we employ the traditional model of McCulloch and Pitts [12] where each neuron is specified by its activity that changes in response to signals arriving from other neurons. A network consists of a set of such elements linked through activatory or inhibitory connections. The dynamics of such a network with arbitrary asymmetric connections is typically characterized by an irregular sequence of complex activity patterns.

In this Letter we consider ensembles made of $N$ identical neural networks each consisting of $K$ neurons. The collective dynamics of an ensemble is described by the

following algorithm: At time $t+1$, the activity $x_k^i$ of a neuron $k = 1, ..., K$ belonging to a network $i = 1, ..., N$ is

$$x_k^i(t+1) = (1 - \varepsilon)\, \Theta(h_k^i) + \varepsilon\, \Theta\left(\sum_{j=1}^{N} h_k^j\right) \qquad (1)$$

where $h_k^i = \sum_{l=1}^{K} J_{kl} x_l^i(t)$ is the signal arriving to this neuron at time $t$ from all other elements of the same network, $J_{kl}$ are the connection weights (the same for all networks), and $\Theta(z)$ is a sigmoidal function, such as $\Theta(z) = 0$ for $z < 0$ and $\Theta(z) = 1$ for $z > 0$.

The two terms in the right side of Eq. (1) have a clear interpretation. The first of them represents the individual response of a neuron to the total signal received from other elements in its own network. The second term depends on the global signal obtained by summation of individual signals received by neurons occupying the same positions in *all* networks of the ensemble. The parameter $\varepsilon$ specifies the strength of global coupling. When global coupling is absent ($\varepsilon = 0$), the networks forming the ensemble are independent. On the other hand, at $\varepsilon = 1$ the first term vanishes and the states of respective neurons in all networks must be identical since they are determined by the same global signal. For $0 < \varepsilon < 1$, the ensemble dynamics is governed by an interplay between local coupling inside the networks and global coupling across them.

Below we show that the model exhibits, under increasing the global coupling intensity, a spontaneous transition to a coherent collective behavior. This transition is characterized by formation of coherent network clusters followed by complete synchronization of all networks in the ensemble. It takes place already at low intensities of global coupling and is observed under an arbitrary choice of the connection weights in elementary networks.

Our analysis is based on numerical investigations. As the first step, we have

set up the connection weights between neurons in the individual network. Each of the connection weights $J_{kl}$ between neurons has been chosen at random with equal probability from the interval between $-1$ to 1. The weights of forward and reverse connections were independently selected, and therefore $J_{kl} \neq J_{lk}$. Most of the simulations were performed for ensembles of $N = 100$ identical networks, each consisting of $K = 50$ neurons. The connection weights remained fixed within the entire series of simulations with varying global coupling intensity. The initial conditions for all neurons in all networks in each simulation have been randomly chosen.

Since subsequent states of all neurons in all networks are recorded, each simulation yields a large volume of data that should be further analyzed in order to detect coherence in the collective activity of the ensemble. An important property is the integral time-dependent activity $u_i(t) = \sum_{k=1}^{K} x_k^i(t)$ of each network $i = 1, ..., N$ in the ensemble. If global coupling is absent or very weak, the networks are independent and, since the initial conditions are various for different networks, their activity patterns are not correlated. Therefore, the integral signals $u_i(t)$ generated by different networks in the ensemble would be asynchronous.

Figure 1 shows typical integral signals generated by networks at higher intensities of global coupling. The initial conditions are chosen at random for each of the networks and therefore the integral signals of the networks are at first not correlated. However, starting from a certain moment, some of the networks in the ensemble begin to generate identical (up to the computer precision) signals, indicating the onset of synchronization in the system. When $\varepsilon = 0.35$ (Fig. 1A), the entire ensemble breaks eventually down into several coherent clusters (the signals generated by networks from the same cluster are displayed here using the same color). At a higher intensity of global coupling ($\varepsilon = 0.5$, Fig. 1B), the activity of *all* networks in

the ensemble becomes eventually coherent. Quite remarkably, the coherent signals are still very complex and apparently chaotic.

The degree of coherence in the ensemble dynamics can be characterized by the dispersion of the activity patterns, defined as $D(t) = N^{-1} \sum_{i=1}^{N} \sum_{k=1}^{K} \left[ x_k^i(t) - \overline{x}_k(t) \right]^2$ with $\overline{x}_k(t) = N^{-1} \sum_{j=1}^{N} x_k^j(t)$. Fig. 2 shows, in logarithmic scale, how this property evolves in time in a typical simulation at a fixed intensity $\varepsilon = 0.5$ of global coupling. Within a transient period, the dispersion $D(t)$ fluctuates but remains, on the average, approximately constant. Then, suddenly, it starts to exponentially decrease with time and rapidly approaches zero. When this occurs, full synchronization of all networks in the ensemble is achieved.

Though the dispersion serves as a good indicator of full synchronization, it is not strongly sensitive to partial synchronization and formation of coherent clusters in the ensemble. To analyze clustering, a different statistical method has therefore been employed which invoved calculation of pair distances between activity patterns of all networks. The pair distance between the activity patterns of two networks $i$ and $j$ is defined as $d_{ij} = \left[ \sum_{k=1}^{K} \left( x_k^i - x_k^j \right)^2 \right]^{1/2}$. By counting the number of network pairs in the whole ensemble that have at a given time the distances lying within subsequent equal intervals, a histogram of distribution over pair distances can be constructed. Fig. 3 shows these normalized histograms for several intensities of global coupling.

When global coupling is weak ($\varepsilon = 0.15$, Fig. 3A), the histogram has a single smooth maximum at a typical distance between the activity patterns of non-correlated networks. Increasing the coupling intensity, we find that above a certain critical point ($\varepsilon_1 \cong 0.17$) the nature of the histogram changes. Now, some pairs of networks in the ensemble have exactly the same activity patterns, so that the distance between them is zero. This corresponds to the presence of a peak at $d = 0$

in Fig. 3B for $\varepsilon = 0.28$. When global coupling is further increased, the number of coherent pairs grows ($\varepsilon = 0.34$, Fig. 3C). Apparently, the coherent networks are already organized into clusters. Indeed, several peaks are clearly seen in this histogram. The peaks are located at pair distances between different coherent clusters. However, besides the coherent clusters the ensemble still has a number of networks with noncoherent activity. A slight increase of global coupling leads to the emergence of a clear cluster organization ($\varepsilon = 0.35$, Fig. 3D). In this case, every network belongs to one of a few coherent clusters. As the coupling intensity grows, the number of clusters gets smaller, until all networks belong to the same coherent group. Full coherence is established in the ensemble starting at $\varepsilon_2 \cong 0.4$ (this final regime is not shown in Fig. 3).

We have repeated our simulations and statistical analysis for different random choices of connection weights in the networks and have observed basically the same sequence of changes leading to clustering and final synchronization in all studied cases, though the respective critical coupling intensities have been found to depend on the choice of connection weights. Moreover, essentially the same results have been obtained when ensembles consisting of larger networks of 100 neurons were studied and when other sigmoidal functions $\Theta(z)$ in the algorythm (1) were employed. This brings us to a suggestion that emergence of coherent collective behavior might be a *generic* property of ensembles formed by globally coupled neural networks.

Full mutual synchronization may play an important role in the action of network ensembles in the brain which are responsible for generation of complex temporal signals and/or determine motor functions. The results reported in this Letter can also be important for studies of animal swarms and robotic populations. Indeed, one can imagine that each of the networks controls the individual behavior of a different population member. Communication between the members may then lead

to global coupling between such networks which can, as we have shown, result in the emergence of perfectly synchronous behaviour or in the breakdown of the population into several coherently operating groups.

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# Figure captions

Fig. 1: Time-dependent integral activity of ten selected networks in an ensemble of $N = 100$ for different intensities of global coupling, corresponding to (A) clustering ($\varepsilon = 0.35$) and (B) complete synchronization ($\varepsilon = 0.5$). Synchronous signals are indicated by the same colors.

Fig. 2: Dispersion of the activity patterns of all networks in the ensemble as function of time under synchronization conditions ($\varepsilon = 0.5$).

Fig. 3: Histograms of distributions over pair distances $d$ between activity patterns of all networks in the ensemble for different intensities of global coupling corresponding to (A) asynchronous regime, $\varepsilon = 0.15$, (B) onset of synchronization, $\varepsilon = 0.28$, (C) partial clustering, $\varepsilon = 0.34$, and (D) full clustering, $\varepsilon = 0.35$.
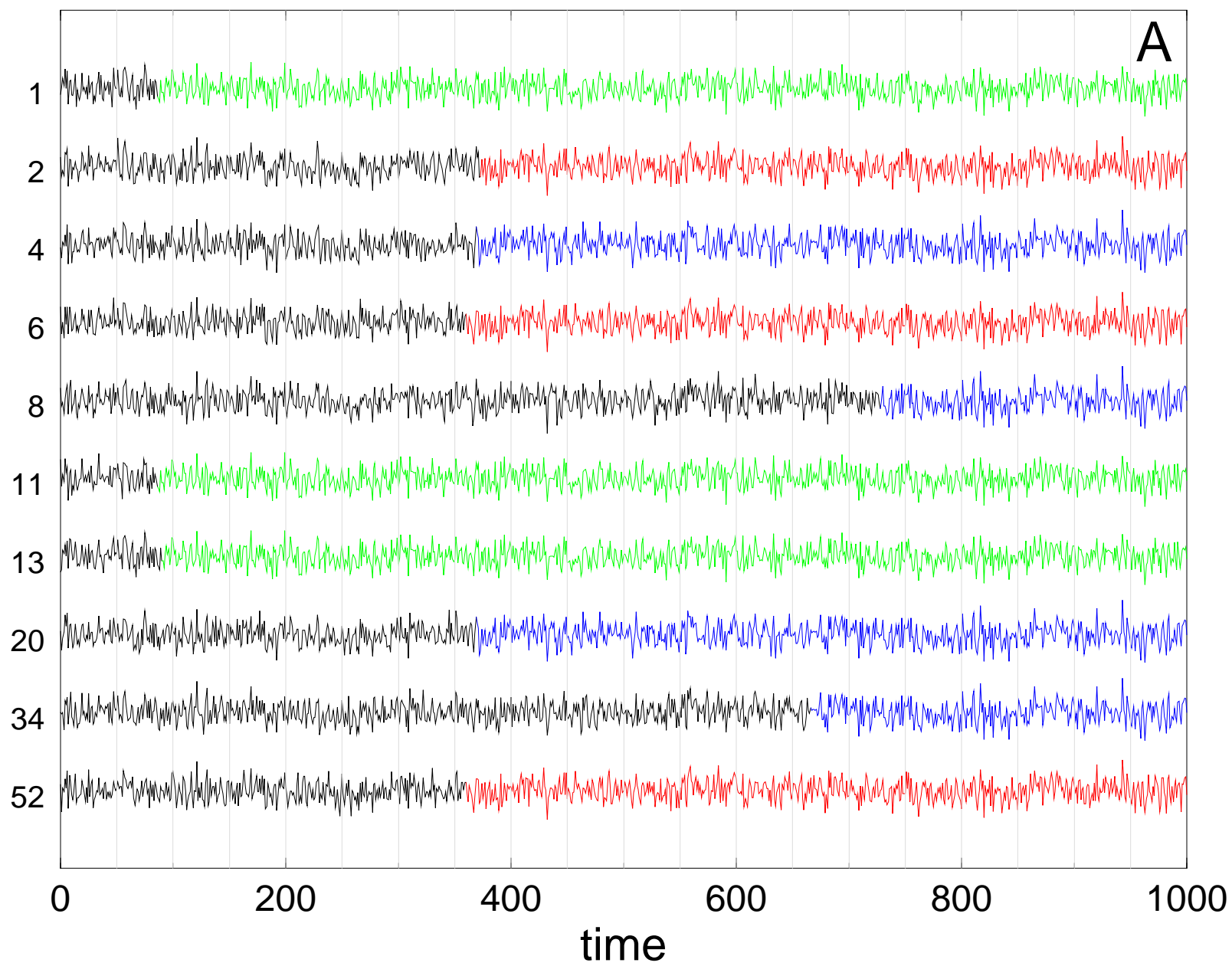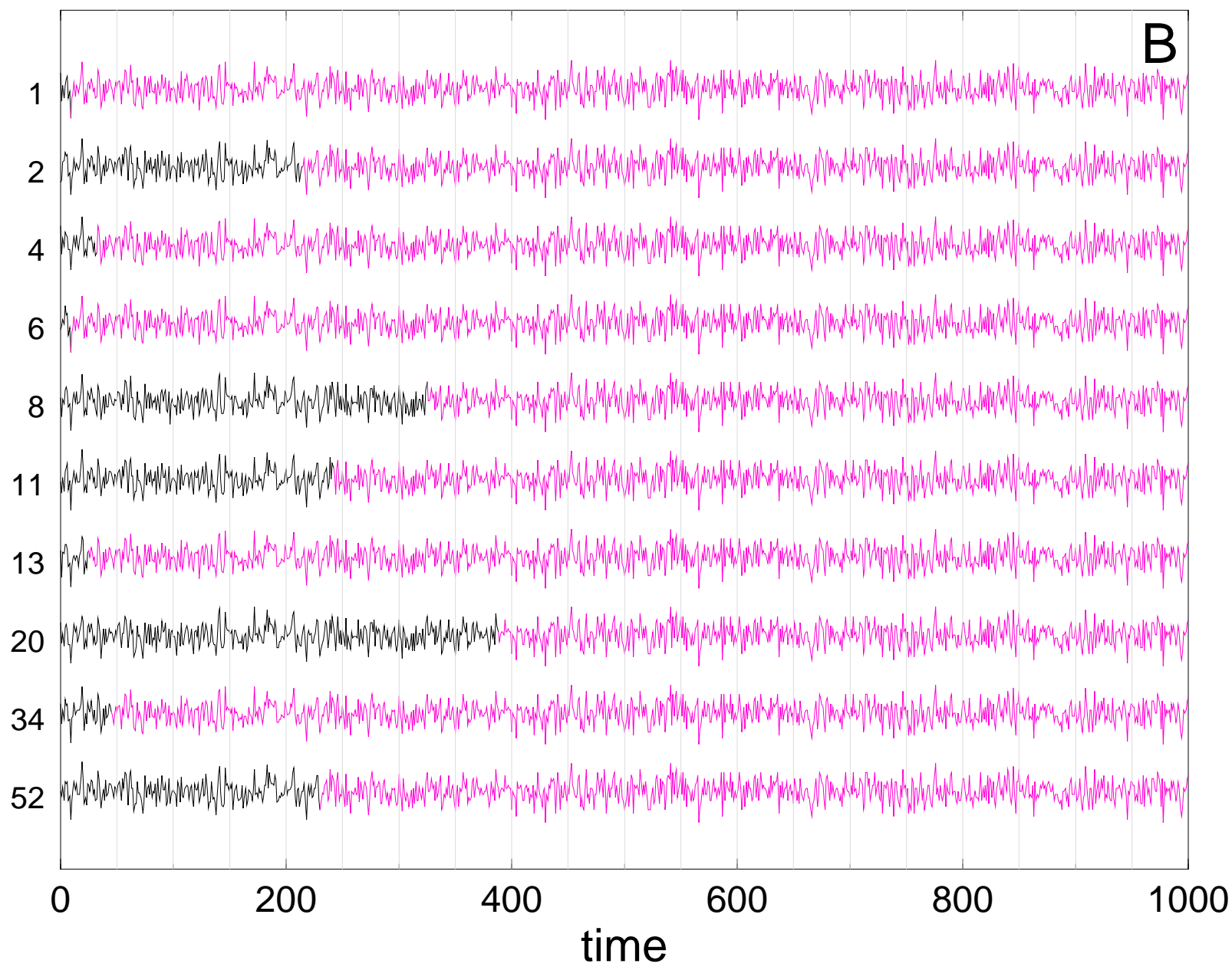
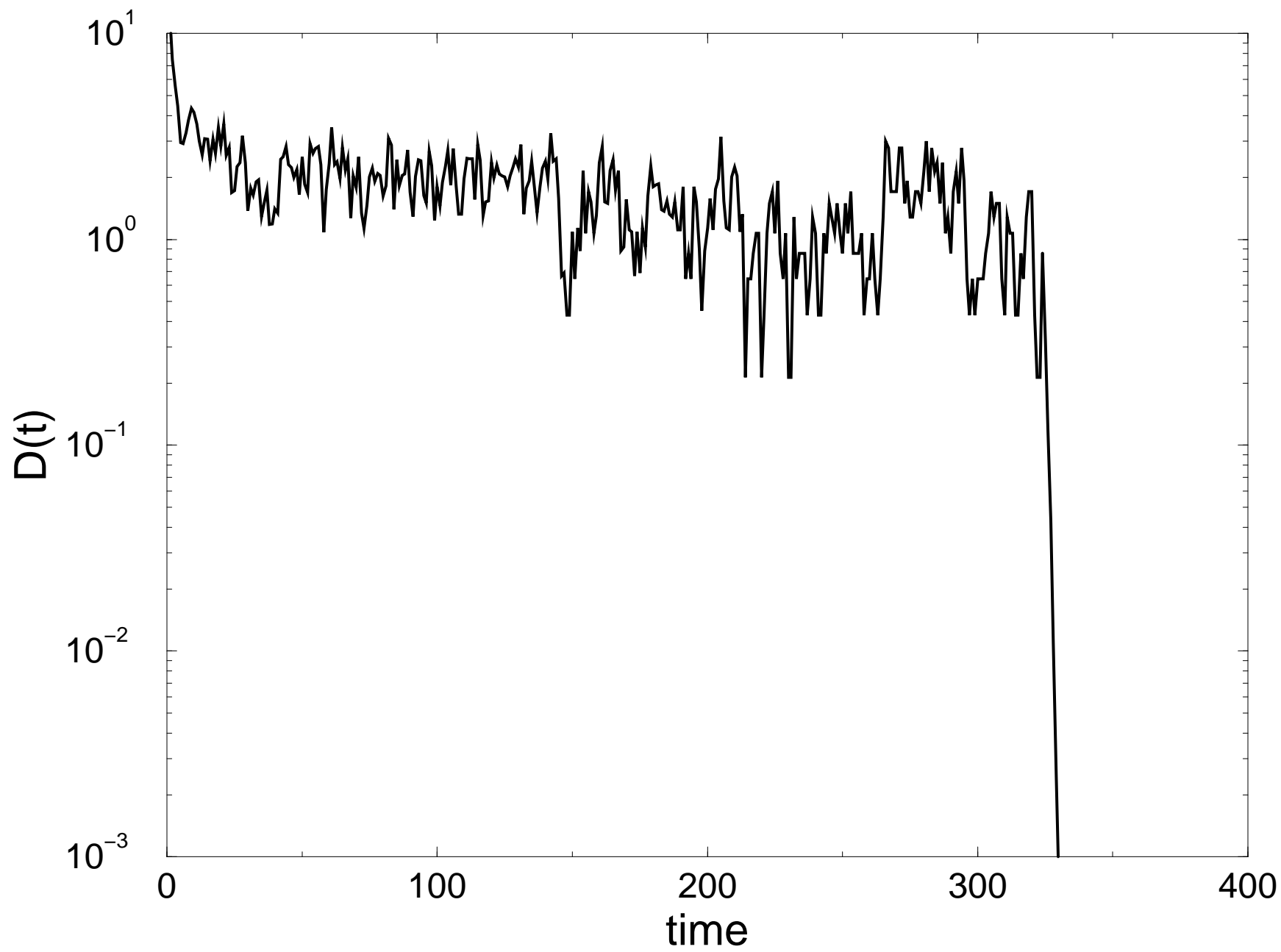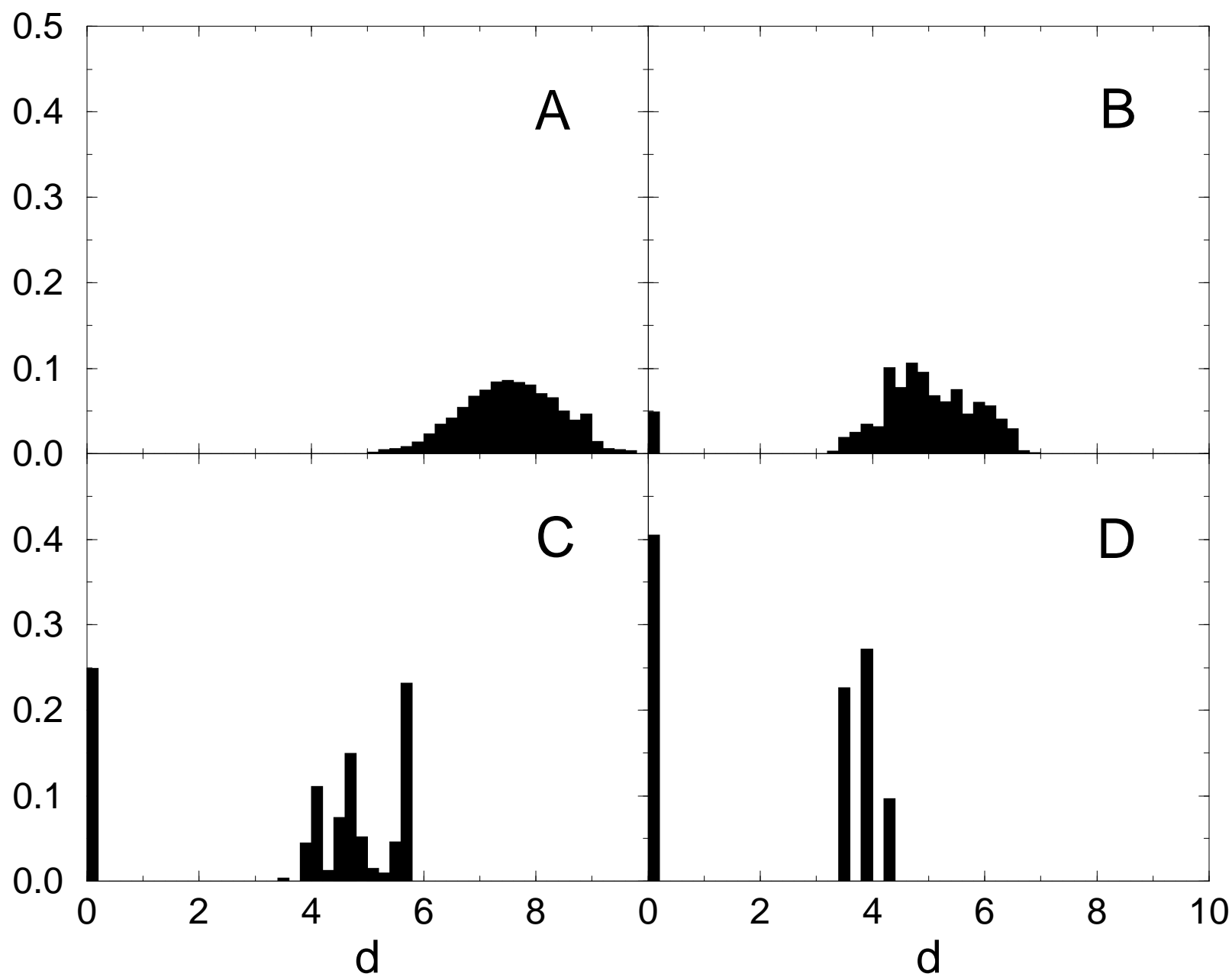Fig. 1A, Zanette & Mikhailov

B

time

Fig. 3, Zanette & Mikhailov